Exercise 5
Momentum expectation value

This exercise guides you through the derivation of the key formula for the expectation value of a particle's momentum $\langle p \rangle$ in terms of its wave function $\Psi(x,t)$. Starting from the definition of $\langle p \rangle$, we have

$$\langle p \rangle \equiv m \frac{d\langle x \rangle}{dt} = m \frac{d}{dt} \int_{-\infty}^{+\infty} x |\Psi|^2 \, dx = m \int_{-\infty}^{+\infty} x \frac{\partial |\Psi|^2}{\partial t} \, dx \, .$$

1. Using the time-dependent Schrödinger equation satisfied by Ψ, show that

$$\frac{\partial |\Psi|^2}{\partial t} = \frac{i\hbar}{2m} \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \, .$$

2. In view of the last equality, we have

$$\langle p \rangle = \frac{i\hbar}{2} \int_{-\infty}^{+\infty} x \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \, dx \, .$$

   By performing integration by parts (twice!), demonstrate the desired result:

$$\langle p \rangle = -i\hbar \int_{-\infty}^{+\infty} \Psi^* \frac{\partial \Psi}{\partial x} \, dx \, .$$

Solution

1. We begin by writing

$$\begin{aligned}\frac{\partial|\Psi|^2}{\partial t} &= \frac{\partial}{\partial t}\Psi^*\Psi \\ &= \frac{\partial\Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial\Psi}{\partial t} \\ &= \left(\frac{\partial\Psi}{\partial t}\right)^*\Psi + \Psi^*\frac{\partial\Psi}{\partial t}\,.\end{aligned}$$

Using the Schrödinger equation

$$\frac{\partial\Psi}{\partial t} = \frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right)\,,$$

we obtain

$$\begin{aligned}\frac{\partial|\Psi|^2}{\partial t} &= -\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right)^*\Psi + \Psi^*\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right) \\ &= -\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + V\Psi^*\right)\Psi + \frac{1}{i\hbar}\Psi^*\left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right) \\ &= \frac{\hbar}{2im}\left(\frac{\partial^2\Psi^*}{\partial x^2}\Psi - \Psi^*\frac{\partial^2\Psi}{\partial x^2}\right)\,.\end{aligned}$$

Then, noticing that

$$\frac{\partial^2\Psi^*}{\partial x^2}\Psi - \Psi^*\frac{\partial^2\Psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\Psi^*}{\partial x}\Psi - \Psi^*\frac{\partial\Psi}{\partial x}\right)\,,$$

we get

$$\frac{\partial|\Psi|^2}{\partial t} = \frac{\hbar}{2im}\frac{\partial}{\partial x}\left(\frac{\partial\Psi^*}{\partial x}\Psi - \Psi^*\frac{\partial\Psi}{\partial x}\right)\,,$$

or, finally,

$$\boxed{\frac{\partial|\Psi|^2}{\partial t} = \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)}$$

2. Doing by parts the integral in

$$\langle p\rangle = \frac{i\hbar}{2}\int_{-\infty}^{+\infty} x\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\,dx\,,$$

we obtain

$$\langle p\rangle = \frac{i\hbar}{2}\left\{ x\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{x\to-\infty}^{x\to+\infty} - \int_{-\infty}^{+\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx\right\}.$$

The wave function and its spatial derivative are assumed to decay rapidly to zero as $x\to\pm\infty$, causing the first term inside the curly brackets to vanish. Hence,

$$\begin{aligned}\langle p\rangle &= -\frac{i\hbar}{2}\int_{-\infty}^{+\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx \\ &= -\frac{i\hbar}{2}\int_{-\infty}^{+\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,dx + \frac{i\hbar}{2}\int_{-\infty}^{+\infty}\frac{\partial\Psi^*}{\partial x}\Psi\,dx\,.\end{aligned}$$

Finally, doing the second integral by parts,

$$\int_{-\infty}^{+\infty}\frac{\partial\Psi^*}{\partial x}\Psi\,dx = \underbrace{\Psi^*\Psi\Big|_{x\to-\infty}^{x\to+\infty}}_{\text{vanishes}} - \int_{-\infty}^{+\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,dx = -\int_{-\infty}^{+\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,dx\,,$$

we arrive at

$$\boxed{\langle p\rangle = -i\hbar\int_{-\infty}^{+\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,dx}$$