Exercise 10
Normalizing a cat state

The wave function of a coherent state of a harmonic oscillator is given by

$$\Psi_{\text{CS}}(x,t;x_0) = \left(\frac{2\alpha}{\pi}\right)^{1/4} \exp\left(-\alpha(x - x_0\cos\omega t)^2 - i2\alpha x x_0 \sin\omega t + i\frac{\alpha x_0^2}{2}\sin 2\omega t - i\frac{\omega t}{2}\right)$$

with

$$\alpha = \frac{m\omega}{2\hbar}\,.$$

It describes a quantum particle that, at time $t = 0$, is released from the position x_0 with zero mean velocity. Here, x and t are the arguments of Ψ_{CS}, while x_0 serves as a parameter.

The superposition

$$\Psi(x,t) = C\Big(\Psi_{\text{CS}}(x,t;x_0) + \Psi_{\text{CS}}(x,t;-x_0)\Big)$$

of two coherent states, one released from x_0 and the other from $-x_0$, is an example of a cat state. Here, C is the normalization constant.

Taking C to be positiove, find its general expression. Then, show that C tends to $1/\sqrt{2}$ as the initial separation between the two coherent states increases.

Solution

Requiring that the normalization integral equals one, we get

$$
\begin{aligned}
1 &= \int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx \\
&= C^2 \int_{-\infty}^{+\infty} \Big(\Psi_{\mathrm{CS}}^*(x,t;x_0) + \Psi_{\mathrm{CS}}^*(x,t;-x_0)\Big)\Big(\Psi_{\mathrm{CS}}(x,t;x_0) + \Psi_{\mathrm{CS}}(x,t;-x_0)\Big)\, dx \\
&= C^2 \Bigg\{ \int_{-\infty}^{+\infty} |\Psi_{\mathrm{CS}}(x,t;x_0)|^2 \, dx + \int_{-\infty}^{+\infty} |\Psi_{\mathrm{CS}}(x,t;-x_0)|^2 \, dx \\
&\qquad + \int_{-\infty}^{+\infty} \Psi_{\mathrm{CS}}^*(x,t;x_0)\Psi_{\mathrm{CS}}(x,t;-x_0)\, dx + \int_{-\infty}^{+\infty} \Psi_{\mathrm{CS}}^*(x,t;-x_0)\Psi_{\mathrm{CS}}(x,t;x_0)\, dx \Bigg\} .
\end{aligned}
$$

The first two integrals inside the curly brackets are the normalization integrals for the two coherent states and are therefore equal to one. The last two integrals are complex conjugates of each other. Hence, we have

$$
C^2(2+I) = 1 \,, \tag{1}
$$

where

$$
I = 2 \operatorname{Re} \int_{-\infty}^{+\infty} \Psi_{\mathrm{CS}}^*(x,t;-x_0)\Psi_{\mathrm{CS}}(x,t;x_0)\, dx
$$

and $\operatorname{Re} z$ represents the real part of z.

It remains to calculate I. We have

$$
\begin{aligned}
I = 2\sqrt{\frac{2\alpha}{\pi}} \operatorname{Re} \int_{-\infty}^{+\infty} \exp\Bigg(& -\alpha\big(x-(-x_0)\cos\omega t\big)^2 + i2\alpha x(-x_0)\sin\omega t - i\frac{\alpha(-x_0)^2}{2}\sin 2\omega t + i\frac{\omega t}{2} \\
& - \alpha(x - x_0\cos\omega t)^2 - i2\alpha x x_0 \sin\omega t + i\frac{\alpha x_0^2}{2}\sin 2\omega t - i\frac{\omega t}{2} \Bigg) dx \\
= 2\sqrt{\frac{2\alpha}{\pi}} \operatorname{Re} \int_{-\infty}^{+\infty} & \exp\Big(-\alpha(x + x_0\cos\omega t)^2 - \alpha(x - x_0\cos\omega t)^2 - i4\alpha x x_0\sin\omega t\Big) dx \\
= 2\sqrt{\frac{2\alpha}{\pi}} \operatorname{Re} \int_{-\infty}^{+\infty} & \exp\Big(-2\alpha x^2 - i4\alpha x x_0 \sin\omega t - 2\alpha x_0^2\cos^2\omega t\Big) dx \,.
\end{aligned}
$$

Using the identity

$$
\int_{-\infty}^{+\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{b^2/4a} \qquad (\operatorname{Re} a > 0)\,,
$$

we get

$$
\begin{aligned}
I &= 2\sqrt{\frac{2\alpha}{\pi}}\,\mathrm{Re}\,\sqrt{\frac{\pi}{2\alpha}}\exp\left(\frac{(-i4\alpha x_0\sin\omega t)^2}{8\alpha} - 2\alpha x_0^2\cos^2\omega t\right)\\
&= 2\exp\Big(-2\alpha x_0^2\left(\sin^2\omega t + \cos^2\omega t\right)\Big)\\
&= 2e^{-2\alpha x_0^2}\,.
\end{aligned}
$$

Then, substituting this expression for I into Eq. (1), we obtain

$$
\boxed{C = \frac{1}{\sqrt{2\left(1+\exp(-2\alpha x_0^2)\right)}}}
$$

It is clear from this result that

$$
C \to \frac{1}{2} \quad \text{as} \quad x_0 \to 0 \quad (\text{coherent states coincide})\,,
$$

and

$$
C \to \frac{1}{\sqrt{2}} \quad \text{as} \quad x_0 \to \infty \quad (\text{coherent states are far apart})\,.
$$