Exercise 1
Coherent state of a harmonic oscillator

Verify that the wave function

$$\Psi(x,t) = C \exp\left(-\alpha(x - x_0 \cos\omega t)^2 - i2\alpha x x_0 \sin\omega t + i\frac{\alpha x_0^2}{2}\sin 2\omega t - i\frac{\omega t}{2}\right)$$

with

$$\alpha = \frac{m\omega}{2\hbar}$$

satisfies the Schrödinger equation for a harmonic oscillator with frequency ω,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{m\omega^2 x^2}{2}\Psi\,.$$

Solution

Substituting Ψ in the left-hand side of the equation, we obtain

$$\begin{aligned}
i\hbar\frac{\partial\Psi}{\partial t} &= i\hbar\left(-2\alpha(x-x_0\cos\omega t)x_0\omega\sin\omega t - i2\alpha x x_0\omega\cos\omega t + i\alpha x_0^2\omega\cos 2\omega t - i\frac{\omega}{2}\right)\Psi \\
&= i\hbar\left(-i2\alpha x x_0\omega(\cos\omega t - i\sin\omega t) + i\alpha x_0^2\omega(\cos 2\omega t - i\sin 2\omega t) - i\frac{\omega}{2}\right)\Psi \\
&= \left(2\alpha x x_0 e^{-i\omega t} - \alpha x_0^2 e^{-i2\omega t} + \frac{1}{2}\right)\hbar\omega\Psi\,. \qquad (1)
\end{aligned}$$

The first term in the right-hand side of the Schrödinger equation reads

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} &= -\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left[-2\alpha(x-x_0\cos\omega t) - i2\alpha x_0\sin\omega t\right]\Psi \\
&= -\frac{\hbar^2}{2m}\left\{\left[-2\alpha(x-x_0\cos\omega t) - i2\alpha x_0\sin\omega t\right]^2 - 2\alpha\right\}\Psi\,.
\end{aligned}$$

Using

$$\frac{\hbar^2}{2m} = \underbrace{\left(\frac{2\hbar}{m\omega}\right)}_{1/\alpha}\frac{\hbar\omega}{4} = \frac{\hbar\omega}{4\alpha}\,,$$

we have

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} &= -\frac{\hbar\omega}{4\alpha}\left\{4\alpha^2\left[x-x_0\cos\omega t + ix_0\sin\omega t\right]^2 - 2\alpha\right\}\Psi \\
&= \left[-\alpha\left(x - x_0e^{-i\omega t}\right)^2 + \frac{1}{2}\right]\hbar\omega\Psi \\
&= \left(-\alpha x^2 + 2\alpha x x_0 e^{-i\omega t} - \alpha x_0^2 e^{-i2\omega t} + \frac{1}{2}\right)\hbar\omega\Psi\,. \qquad (2)
\end{aligned}$$

The second term in the right-hand side of the Schrödinger equation can be written as

$$\frac{m\omega^2x^2}{2}\Psi = \alpha\hbar\omega x^2\Psi\,. \qquad (3)$$

Comparing Eqs. (1), (2) and (3), we conclude that

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{m\omega^2x^2}{2}\Psi\,.$$