Exercise 8
An example of Heisenberg's uncertainty principle

At time $t = 0$, a particle is described by the wave function

$$\Psi(x,0) = \begin{cases} Cx(a-x) & \text{if} \quad 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

where $a > 0$ is a parameter, and C is the normalization constant.

1. Express C in terms of a. Then, sketch the wave function and the corresponding probability density.

2. Determine the uncertainties in the particle's position and momentum. Then, verify that the product of these uncertainties exceeds $\hbar/2$, thereby confirming the Heisenberg uncertainty principle.

Solution

1. Equating the normalization integral to one, we obtain

$$\begin{aligned}
1 &= \int_{-\infty}^{+\infty} |\Psi(x,0)|^2 \, dx \\
&= C^2 \int_0^a x^2 (a-x)^2 \, dx \\
&= C^2 \left(a^2 \int_0^a x^2 \, dx - 2a \int_0^a x^3 \, dx + \int_0^a x^4 \, dx \right) \\
&= C^2 a^5 \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) \\
&= C^2 \frac{a^5}{30} \, .
\end{aligned}$$

Hence, the normalization constant is given by

$$\boxed{C = \sqrt{\frac{30}{a^5}}}$$

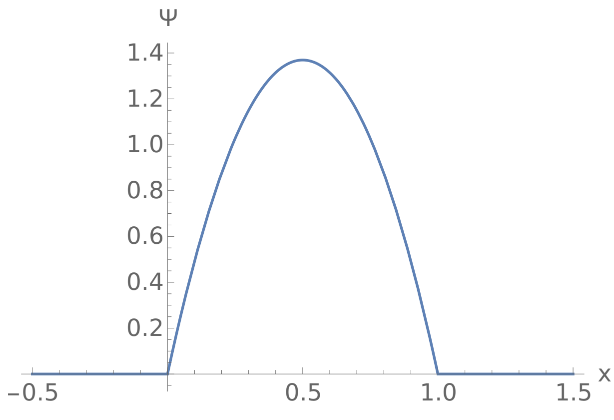

Figure 1: Wave function $\Psi(x,0)$ for $a = 1$.

Having determined the normalization constant, we now plot the wave function $\Psi(x,0)$ and probability density $|\Psi(x,0)|^2$ in Figures 1 and 2, respectively.

2. The probability density $|\Psi(x,0)|^2$ is symmetric about $x = a/2$. Therefore, the expectation value of the particle's position is given by (see Exercise 3)

$$\boxed{\langle x \rangle = \frac{a}{2}}$$

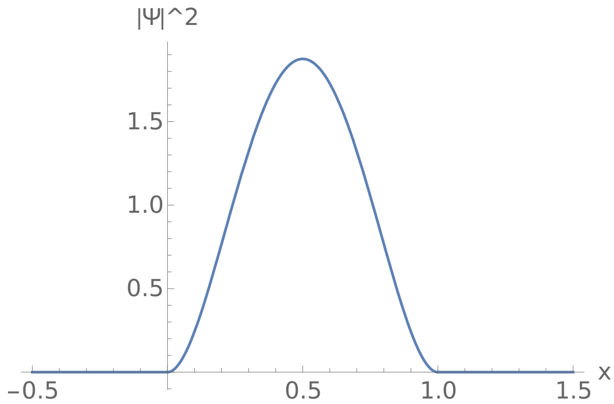

Figure 2: Probability density $|\Psi(x,0)|^2$ for $a = 1$.

The wave function $\Psi(x,0)$ is real. Therefore, the expectation value of the particle's momentum is zero (see Exercise 6):

$$\boxed{\langle p \rangle = 0}$$

Now, let's calculate $\langle x^2 \rangle$ and $\langle p^2 \rangle$. We have

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{+\infty} x^2 |\Psi(x,0)|^2 \, dx \\
&= C^2 \int_0^a x^4 (a-x)^2 \, dx \\
&= C^2 \left(a^2 \int_0^a x^4 \, dx - 2a \int_0^a x^5 \, dx + \int_0^a x^6 \, dx \right) \\
&= C^2 a^7 \left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7} \right) \\
&= \frac{30}{a^5} a^7 \frac{1}{105} \,,
\end{aligned}$$

$$\boxed{\langle x^2 \rangle = \frac{2a^2}{7}}$$

and

$$\begin{aligned}\langle p^2\rangle &= \int_{-\infty}^{+\infty} \Psi^*(x,0)\left(-i\hbar\frac{\partial}{\partial x}\right)^2 \Psi(x,0)\,dx \\ &= -\hbar^2\int_{-\infty}^{+\infty} \Psi^*(x,0)\frac{\partial^2\Psi(x,0)}{\partial x^2}\,dx \\ &= -\hbar^2\int_0^a Cx(a-x)(-2C)\,dx \\ &= 2C^2\hbar^2\left(a\int_0^a x\,dx - \int_0^a x^2\,dx\right) \\ &= 2C^2\hbar^2a^3\left(\frac{1}{2}-\frac{1}{3}\right) \\ &= 2\frac{30}{a^5}\hbar^2a^3\frac{1}{6}\,,\end{aligned}$$

$$\boxed{\langle p^2\rangle = \frac{10\hbar^2}{a^2}}$$

We can now calculate the uncertainties in the particle's position and momentum. For the position uncertainty, we have

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{2a^2}{7} - \left(\frac{a}{2}\right)^2\,,$$

or

$$\boxed{\Delta x = \frac{a}{2\sqrt{7}}}$$

For the momentum uncertainty, we have

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{10\hbar^2}{a^2} - 0\,,$$

or

$$\boxed{\Delta p = \frac{\sqrt{10}\hbar}{a}}$$

The product of the uncertainties exceeds the Heisenberg threshold of $\hbar/2$. Indeed,

$$\Delta x\Delta p = \frac{a}{2\sqrt{7}}\frac{\sqrt{10}\hbar}{a} = \sqrt{\frac{10}{7}}\frac{\hbar}{2} > \frac{\hbar}{2}\,.$$