Exercise 9
Derivation of Heisenberg's uncertainty principle

Consider a quantum particle characterized by a wave function $\Psi(x,t)$, and define the following non-negative real-valued function:

$$f(u) = \int_{-\infty}^{+\infty} \left| u\left(\hat{x} - \langle x \rangle\right)\Psi + i\left(\hat{p} - \langle p \rangle\right)\Psi \right|^2 dx\,.$$

Here, $\hat{x} = x$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ are the position and momentum operators, respectively, and $\langle x \rangle$ and $\langle p \rangle$ are the corresponding expectation values.

1. Show that the function $f(u)$ can be written as

   $$f(u) = (\Delta x)^2 u^2 - \hbar u + (\Delta p)^2\,,$$

   where $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$ are the uncertainties in the particle's position and momentum.

2. Then, show that the non-negativity of $f(u)$, i.e. the condition that

   $$f(u) \geq 0 \quad \text{for all } u\,,$$

   implies that Δx and Δp fulfil the Heisenberg uncertainty relation,

   $$\Delta x \Delta p \geq \frac{\hbar}{2}\,.$$

Solution

1. The integrand can be transformed as

$$
\begin{aligned}
&\left|u\left(\hat{x}-\langle x\rangle\right)\Psi+i\left(\hat{p}-\langle p\rangle\right)\Psi\right|^2\\
&=\left|u\left(x-\langle x\rangle\right)\Psi+i\left(-i\hbar\frac{\partial}{\partial x}-\langle p\rangle\right)\Psi\right|^2\\
&=\left|u\left(x-\langle x\rangle\right)\Psi+\hbar\frac{\partial\Psi}{\partial x}-i\langle p\rangle\Psi\right|^2\\
&=\left(u\left(x-\langle x\rangle\right)\Psi^*+\hbar\frac{\partial\Psi^*}{\partial x}+i\langle p\rangle\Psi^*\right)\left(u\left(x-\langle x\rangle\right)\Psi+\hbar\frac{\partial\Psi}{\partial x}-i\langle p\rangle\Psi\right)\\
&=u^2\left(x-\langle x\rangle\right)^2|\Psi|^2+\left|\hbar\frac{\partial\Psi}{\partial x}-i\langle p\rangle\Psi\right|^2\\
&\qquad+u\left(x-\langle x\rangle\right)\Psi^*\left(\hbar\frac{\partial\Psi}{\partial x}-\cancel{i\langle p\rangle\Psi}\right)+\left(\hbar\frac{\partial\Psi^*}{\partial x}+\cancel{i\langle p\rangle\Psi^*}\right)u\left(x-\langle x\rangle\right)\Psi\\
&=u^2\left(x-\langle x\rangle\right)^2|\Psi|^2+\left|\hbar\frac{\partial\Psi}{\partial x}-i\langle p\rangle\Psi\right|^2+u\hbar\left(x-\langle x\rangle\right)\frac{\partial|\Psi|^2}{\partial x}\,.
\end{aligned}
$$

Then, for the function $f(u)$, we have

$$
\begin{aligned}
f(u)=u^2&\int_{-\infty}^{+\infty}\left(x-\langle x\rangle\right)^2|\Psi|^2dx\\
&+u\hbar\int_{-\infty}^{+\infty}\left(x-\langle x\rangle\right)\frac{\partial|\Psi|^2}{\partial x}dx\\
&+\int_{-\infty}^{+\infty}\left|\hbar\frac{\partial\Psi}{\partial x}-i\langle p\rangle\Psi\right|^2dx\,.
\end{aligned}
\tag{1}
$$

Clearly, the first integral in Eq. (1) is nothing but $(\Delta x)^2$:

$$
\int_{-\infty}^{+\infty}\left(x-\langle x\rangle\right)^2|\Psi|^2dx=(\Delta x)^2\,.
$$

The second integral in Eq. (1) is evaluated using integration by parts and the normalization condition:

$$
\int_{-\infty}^{+\infty}\left(x-\langle x\rangle\right)\frac{\partial|\Psi|^2}{\partial x}dx=-\int_{-\infty}^{+\infty}\frac{\partial\left(x-\langle x\rangle\right)}{\partial x}|\Psi|^2dx=-\int_{-\infty}^{+\infty}|\Psi|^2dx=-1\,.
$$

The first integral in Eq. (1) equals $(\Delta p)^2$. Indeed,

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left| \hbar \frac{\partial \Psi}{\partial x} - i\langle p \rangle \Psi \right|^2 dx \\
&= \int_{-\infty}^{+\infty} \left(\hbar \frac{\partial \Psi^*}{\partial x} + i\langle p \rangle \Psi^* \right) \left(\hbar \frac{\partial \Psi}{\partial x} - i\langle p \rangle \Psi \right) dx \\
&= \int_{-\infty}^{+\infty} \hbar \frac{\partial \Psi^*}{\partial x} \left(\hbar \frac{\partial \Psi}{\partial x} - i\langle p \rangle \Psi \right) dx + \int_{-\infty}^{+\infty} i\langle p \rangle \Psi^* \left(\hbar \frac{\partial \Psi}{\partial x} - i\langle p \rangle \Psi \right) dx \\
&= \int_{-\infty}^{+\infty} (-\hbar) \Psi^* \frac{\partial}{\partial x} \left(\hbar \frac{\partial \Psi}{\partial x} - i\langle p \rangle \Psi \right) dx + \int_{-\infty}^{+\infty} i\langle p \rangle \Psi^* \left(\hbar \frac{\partial \Psi}{\partial x} - i\langle p \rangle \Psi \right) dx \\
&= \int_{-\infty}^{+\infty} \Psi^* \left(-\hbar \frac{\partial}{\partial x} + i\langle p \rangle \right) \left(\hbar \frac{\partial \Psi}{\partial x} - i\langle p \rangle \Psi \right) dx \\
&= \int_{-\infty}^{+\infty} \Psi^* \left(-i\hbar \frac{\partial}{\partial x} - \langle p \rangle \right) \left(-i\hbar \frac{\partial \Psi}{\partial x} - \langle p \rangle \Psi \right) dx \\
&= \int_{-\infty}^{+\infty} \Psi^* \left(\hat{p} - \langle p \rangle \right)^2 \Psi \, dx \\
&= (\Delta p)^2 \,.
\end{aligned}$$

Finally, replacing the three integrals in Eq. (1) with $(\Delta x)^2$, -1 and $(\Delta p)^2$, respectively, we obtain

$$\boxed{f(u) = (\Delta x)^2 u^2 - \hbar u + (\Delta p)^2}$$

2. Competing the full square,

$$\begin{aligned}
f(u) &= (\Delta x)^2 u^2 - \hbar u + (\Delta p)^2 \\
&= (\Delta x)^2 \left(u - \frac{\hbar}{2(\Delta x)^2} \right)^2 + (\Delta p)^2 - \frac{\hbar^2}{4(\Delta x)^2} \,.
\end{aligned}$$

This expression is non-negative for all u if and only if

$$(\Delta p)^2 - \frac{\hbar^2}{4(\Delta x)^2} \geq 0 \,.$$

Since Δx and Δp are themselves positive, the last inequality is equivalent to the Heisenberg uncertainty principle,

$$\boxed{\Delta x \Delta p \geq \frac{\hbar}{2}}$$