EXERCISE 17
Energy measurement for a particle in a box

A particle of mass m is confined between two impenetrable walls, one located at $x = 0$ and the other at $x = L$. At time $t = 0$, the wave function of the particle is given by

$$\Psi(x, 0) = C \sin\left(\frac{\pi x}{L}\right)\left[1 + i\cos\left(\frac{\pi x}{L}\right)\right] ,$$

where $C > 0$ is the normalization constant.

1. Determine the normalization constant.
2. Calculate the expectation value of the particle's energy.
3. If an energy measurement is performed on the system, what is the probability that the measurement will yield the ground state energy? Additionally, what will the wave function of the particle be immediately after the measurement?
4. What is the probability that the energy measurement will return a value equal to twice the ground state energy?
5. What is the probability that the measurement will yield the energy of the second excited state?

Solution

1. There are two ways of determining the normalization constant.

   (a) One approach is to realize that $\Psi(x,0)$ is a superposition of only two stationary states – the ground state and the first excited state – of the particle-in-a-box system. Indeed,

   $$\begin{aligned}\Psi(x,0) &= C\sin\left(\frac{\pi x}{L}\right) + iC\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)\\ &= C\sin\left(\frac{\pi x}{L}\right) + i\frac{C}{2}\sin\left(\frac{2\pi x}{L}\right)\end{aligned}$$

   and, therefore,

   $$\Psi(x,0) = c_1\psi_1(x) + c_2\psi_2(x)\,, \tag{1}$$

   where

   $$\psi_1(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) \quad \text{and} \quad \psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$

   are the ground state and the first excited state, respectively, and

   $$c_1 = C\sqrt{\frac{L}{2}} \quad \text{and} \quad c_2 = i\frac{C}{2}\sqrt{\frac{L}{2}} \tag{2}$$

   are the corresponding expansion coefficients.
   Then, substituting Eq. (2) into the normalization condition (see Exercise 15),

   $$|c_1|^2 + |c_2|^2 = 1\,,$$

   we obtain

   $$C^2\frac{L}{2}\left(1+\frac{1}{4}\right) = 1\,,$$

   and, subsequently,

   $$\boxed{C = 2\sqrt{\frac{2}{5L}}}$$

   (b) The other approach is to directly evaluate the normalization integral and equate it to one:

   $$\int_0^L |\Psi(x,0)|^2\,dx = 1\,.$$

   This calculation produces the same result as the previous method.

2. The simplest way to calculate the expectation value of the particle's energy is by using the formula derived in Exercise 16:

$$\langle H \rangle = \sum_n E_n |c_n|^2 .$$

Using Eq. (2), along with the value of the normalization constant, we have

$$c_1 = \frac{2}{\sqrt{5}} \quad \text{and} \quad c_2 = \frac{i}{\sqrt{5}} .$$

Then,

$$\begin{aligned} \langle H \rangle &= E_1 |c_1|^2 + E_2 |c_2|^2 \\ &= \frac{\pi^2 \hbar^2}{2mL^2} \left| \frac{2}{\sqrt{5}} \right|^2 + \frac{4\pi^2 \hbar^2}{2mL^2} \left| \frac{i}{\sqrt{5}} \right|^2 \\ &= \frac{\pi^2 \hbar^2}{2mL^2} \left(\frac{4}{5} + 4 \cdot \frac{1}{5} \right) , \end{aligned}$$

or

$$\boxed{\langle H \rangle = \frac{4\pi^2 \hbar^2}{5mL^2}}$$

3. The wave function of the system right before the energy measurement is given by Eq. (1). The probability for the measurement to yield the ground state energy, E_1, is

$$P(E = E_1) = |c_1|^2 = \left| \frac{2}{\sqrt{5}} \right|^2 = \frac{4}{5} = 80\% .$$

The measurement that returns the energy value E_1 "collapses" the system's wave function onto the corresponding stationary state, which in this case is $\psi_1(x)$. So, the wave function of the particle immediately after this measurement will be $\psi_1(x)$.

4. An energy measurement can only return the energy value of one of the stationary states. Since there is no stationary state with energy $2E_1$ (indeed, the value $2E_1$ lies between the ground state energy, E_1, and the energy of the first excited state, $E_2 = 4E_1$) the probability of an energy measurement yielding twice the ground state energy is zero.

5. Generally, the probability of an energy measurement returning E_3 (the energy of the second excited state) is $|c_3|^2$. For the wave function given by Eq. (1), $c_3 = 0$, and so this probability is zero.