# Exercise 20
## Step potential:
## Probabilities of reflection and transmission

In Exercise 19, you demonstrated that a particle with energy $E$ scattering at a potential step of height $V_0$ is described by the function $\psi(x)$, which in the incident region ($x < 0$) takes the following form:

$$\psi(x) = e^{ikx} + Be^{-ikx}\,.$$

The reflection amplitude $B$ was also calculated in that exercise.

The probability $R$ that the particle is reflected by the potential step is given by the squared modulus of the reflection amplitude $B$, similar to how the probability of finding a particle near a specific point is proportional to the squared modulus of the particle's wave function at that point. Thus,

$$R = |B|^2\,.$$

If the particle is not reflected, it will be transmitted over the potential step. Therefore, the transmission probability $T$ is given by

$$\begin{aligned} T &= 1 - R \\ &= 1 - |B|^2\,. \end{aligned}$$

1. Calculate and sketch the reflection and transmission probabilities, $R$ and $T$, as functions of the particle's energy $E$.
2. Then, show that as $V_0$ tends to negative infinity, representing a sudden, infinitely deep potential drop, $R$ approaches unity. This implies that the particle is completely reflected by an infinitely deep potential drop, illustrating the counterintuitive phenomenon of quantum reflection.

# Solution

1. The amplitude of the reflected plane wave has been determined in Exercise 19. It is given by

$$B = \begin{cases} \frac{k-ik'}{k+ik'} & \text{if} \quad E < V_0 \\ \frac{k-k''}{k+k''} & \text{if} \quad E > V_0 \end{cases} ,$$

where

$$k = \frac{\sqrt{2mE}}{\hbar} , \quad k' = \frac{\sqrt{2m(V_0 - E)}}{\hbar} , \quad k'' = \frac{\sqrt{2m(E - V_0)}}{\hbar} .$$

Expressed directly as a function of $E$, it reads

$$B = \begin{cases} \frac{\sqrt{E}-i\sqrt{V_0-E}}{\sqrt{E}+i\sqrt{V_0-E}} & \text{if} \quad E < V_0 \\ \frac{\sqrt{E}-\sqrt{E-V_0}}{\sqrt{E}+\sqrt{E-V_0}} & \text{if} \quad E > V_0 \end{cases} .$$

Then, the reflection probability is given by

$$R = |B|^2 = \begin{cases} 1 & \text{if} \quad E < V_0 \\ \left(\frac{\sqrt{E}-\sqrt{E-V_0}}{\sqrt{E}+\sqrt{E-V_0}}\right)^2 & \text{if} \quad E > V_0 \end{cases} . \tag{1}$$

Here, we have used the fact that

$$\left| \frac{\sqrt{E} - i\sqrt{V_0 - E}}{\sqrt{E} + i\sqrt{V_0 - E}} \right| = 1 ,$$

which holds true since $|z^*/z| = 1$ for any non-zero complex number $z$.

Subsequently, the transmission probability is given by

$$T = 1 - R = \begin{cases} 0 & \text{if} \quad E < V_0 \\ \frac{4\sqrt{E(E-V_0)}}{\left(\sqrt{E}+\sqrt{E-V_0}\right)^2} & \text{if} \quad E > V_0 \end{cases} . \tag{2}$$

Figures 1 and 2 show the plots of the reflection and transmission probabilities computed using Eqs. (1) and (2), respectively. It is clear from the plots (and the corresponding equations) that the particle undergoes compete reflection from the potential step when its energy is lower than the step height.

2. Let's consider the scenario of a negative potential step. That is, let's take $V_0 < 0$, so that

$$V_0 = -|V_0| .$$

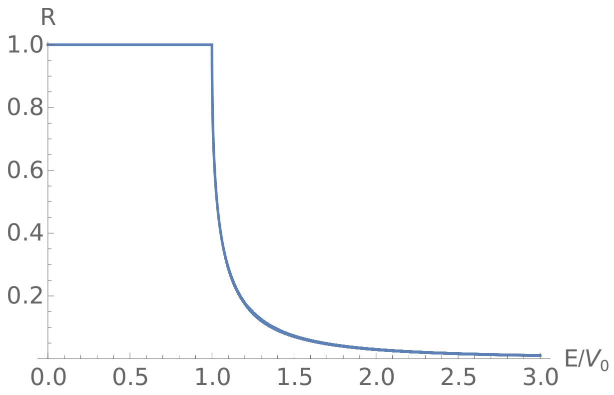


Figure 1: Reflection probability $R$ as a function of $E/V_0$.

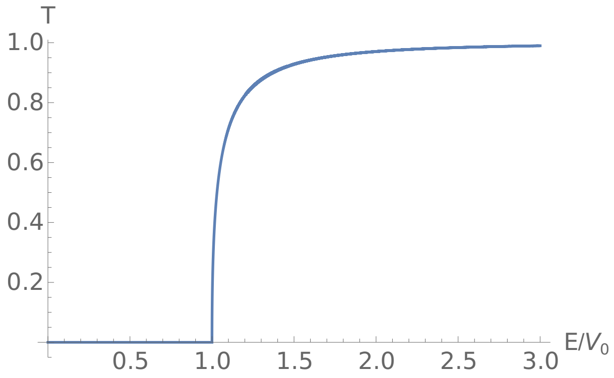


Figure 2: Transmission probability $T$ as a function of $E/V_0$.

Since the particle's energy is positive, $E > 0$, we find ourselves in the $E > V_0$ regime. In this regime, the reflection probability is given by

$$\begin{aligned} R &= \left(\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}}\right)^2 \\ &= \left(\frac{\sqrt{E} - \sqrt{E + |V_0|}}{\sqrt{E} + \sqrt{E + |V_0|}}\right)^2 . \end{aligned}$$

As $V_0$ tends to minus infinity, $\sqrt{E + |V_0|}$ becomes infinitely large, rendering the term $\sqrt{E}$ negligible in comparison. Therefore, as $V_0 \to -\infty$, the reflection probability tends to one:

$$R \to \left(\frac{-\sqrt{E + |V_0|}}{\sqrt{E + |V_0|}}\right)^2 = 1 .$$

This means that a quantum particle incident upon an infinitely deep and steep potential drop gets completely reflected, rather than falling down the drop as one might expect based on classical mechanical intuition. This is the essence of the phenomenon of quantum reflection.