Exercise 11
Normalization constant for the stationary states of a particle in a box

The spatial part of the n^{th} stationary state wave function of a particle in a box is given by

$$\psi_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) ,$$

where L is the width of the box and B_n is the normalization constant.

Assuming B_n is positive, show that it is given by

$$B_n = \sqrt{\frac{2}{L}}$$

and is therefore independent of n.

Solution

The normalization integral can be calculated as follows:

$$\begin{aligned}
\int_0^L |\psi_n(x)|^2\,dx &= B_n^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\,dx \\
&= B_n^2 \int_0^L \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{L}\right)\right]\,dx \\
&= \frac{B_n^2}{2}\int_0^L dx - \frac{B_n^2}{2}\int_0^L \cos\left(\frac{2n\pi x}{L}\right)\,dx \\
&= \frac{B_n^2}{2}L - \frac{B_n^2}{2}\frac{L}{2n\pi}\underbrace{\sin\left(\frac{2n\pi x}{L}\right)\Bigg|_{x=0}^{x=L}}_{0} \\
&= B_n^2\frac{L}{2}\,.
\end{aligned}$$

Equating the normalization integral to one, we find that

$$\boxed{B_n = \sqrt{\frac{2}{L}}}$$

So, the normalization constant is indeed independent of the quantum number n.