Exercise 4
Position expectation value: An example

At time $t = 0$, the wave function of a particle is given by

$$\Psi(x,0) = \begin{cases} C(x - \beta x^2)e^{-\alpha x} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases} .$$

Here, α and β are positive constants with dimensions of inverse length, and C is the normalization constant.

1. Determine the normalization constant C in terms of α and β.
2. Sketch the wave function Ψ and the corresponding probability density $|\Psi|^2$.
3. Calculate the expectation value of the particle's position, $\langle x \rangle$. Express $\langle x \rangle$ as a function of α and β.
4. Now, consider the special case where $\beta = 0.549778 \times \alpha$. In this situation, you can easily verify that the wave function almost exactly vanishes at $x = \langle x \rangle$. This implies that it is highly improbable to find the particle near this "expected position". Discuss the implications of this observation. What does this suggest about the expectation value as a measure of the particle's position?

Solution

1. The probability density corresponding to $\Psi(x,0)$ is given by

$$|\Psi(x,0)|^2 = \begin{cases} C^2(x^2 - 2\beta x^3 + \beta^2 x^4)e^{-2\alpha x} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases} .$$

The normalization constant C is obtained by equating the normalization integral to one:

$$\begin{aligned} 1 &= \int_0^\infty |\Psi(x,0)|^2 \, dx \\ &= C^2 \int_0^\infty x^2 e^{-2\alpha x} \, dx - 2\beta C^2 \int_0^\infty x^3 e^{-2\alpha x} \, dx + \beta^2 C^2 \int_0^\infty x^4 e^{-2\alpha x} \, dx \, . \end{aligned}$$

In view of the integral

$$\int_0^\infty x^n e^{-2\alpha x} \, dx = \frac{n!}{(2\alpha)^{n+1}} \, ,$$

the equation for C becomes

$$1 = C^2 \left(\frac{2}{(2\alpha)^3} - 2\beta \frac{3 \cdot 2}{(2\alpha)^4} + \beta^2 \frac{4 \cdot 3 \cdot 2}{(2\alpha)^5} \right) = \frac{C^2}{4\alpha^5}(\alpha^2 - 3\alpha\beta + 3\beta^2) \, .$$

Therefore,

$$\boxed{C = \frac{2\alpha^{5/2}}{\sqrt{\alpha^2 - 3\alpha\beta + 3\beta^2}}}$$

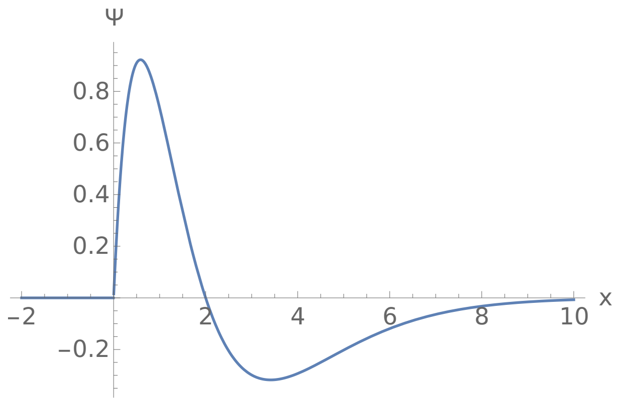

Figure 1: Wave function $\Psi(x,0)$ for $\alpha = 1$ and $\beta = 0.5$.

2. Having determined the normalization constant we can now plot the wave function and probability density. Figures 1 and 2 show the wave function $\Psi(x,0)$ and probability density $|\Psi(x,0)|^2$, respectively, for the case where $\alpha = 1$ and $\beta = 0.5$. Note

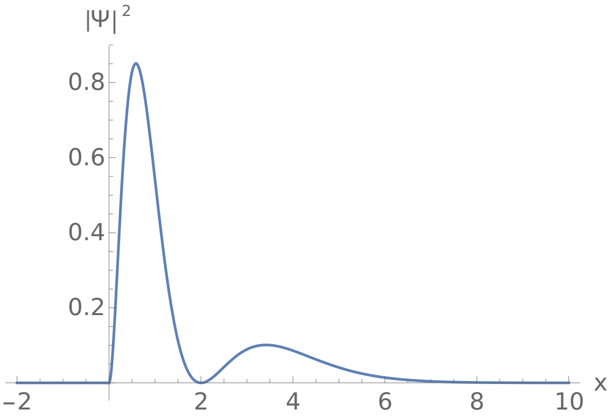

Figure 2: Probability density $|\Psi(x,0)|^2$ for $\alpha = 1$ and $\beta = 0.5$.

that the probability density has two peaks and turns exactly zero at the point $x = 1/\beta$, which lies between the peaks.

3. The expectation value of the particle's position is calculated as follows:

$$\begin{aligned}\langle x\rangle &= \int_0^\infty x|\Psi(x,0)|^2\,dx\\ &= C^2\int_0^\infty x^3 e^{-2\alpha x}\,dx - 2\beta C^2\int_0^\infty x^4 e^{-2\alpha x}\,dx + \beta^2 C^2\int_0^\infty x^5 e^{-2\alpha x}\,dx\\ &= C^2\left(\frac{3\cdot 2}{(2\alpha)^4} - 2\beta\frac{4\cdot 3\cdot 2}{(2\alpha)^5} + \beta^2\frac{5\cdot 4\cdot 3\cdot 2}{(2\alpha)^6}\right)\\ &= \frac{3C^2}{8\alpha^6}(\alpha^2 - 4\alpha\beta + 5\beta^2)\,.\end{aligned}$$

Using the previously obtained expression for the normalization constant, we arrive at

$$\boxed{\langle x\rangle = \frac{3(\alpha^2 - 4\alpha\beta + 5\beta^2)}{2\alpha(\alpha^2 - 3\alpha\beta + 3\beta^2)}}$$

4. Substituting

$$\beta = 0.549778\times\alpha$$

into the above expression for the position expectation value, we find that

$$\langle x\rangle = \frac{1.81891938147548 72}{\alpha} = \frac{1.0000018597088303}{\beta} \simeq \frac{1}{\beta}\,.$$

Although the last equality is an approximation, it is a very accurate one: the relative error is around 2×10^{-6} (or 0.0002%).

Recall now that $\Psi(x,0)$ vanishes at $x = 1/\beta$. This means that both the wave function and the probability density nearly vanish at the expectation value of the particle's position, i.e., at $x = \langle x \rangle$. Consequently, if one measures the position of the particle using a narrow detector, one will almost never find the particle in the vicinity of its position expectation value!

However, there is no true contradiction in this seemingly paradoxical result. The term "expectation value" is simply a (somewhat misleading) label for the average position obtained from an infinite number of measurements on identical systems. It does not necessarily indicate the position where the particle is most likely to be found in a single measurement.