Exercise 3
Position expectation value in symmetric distributions

Consider a particle described by the probability density $|\Psi(x,t)|^2$. Suppose that $|\Psi(x,t)|^2$ is symmetric about $x = a$, where a may be a function of time t. That is,

$$|\Psi(a-x,t)|^2 = |\Psi(a+x,t)|^2$$

for all x. Show that the expectation value of the particle's position is equal to a, i.e.

$$\langle x \rangle = a\,.$$

Solution

The calculation of the expectation value of the particle's position proceeds as follows:

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{+\infty} x|\Psi(x,t)|^2\,dx \\ &= \int_{-\infty}^{+\infty} [a+(x-a)]|\Psi(x,t)|^2\,dx \\ &= a\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\,dx + \int_{-\infty}^{+\infty} (x-a)|\Psi(x,t)|^2\,dx\,.\end{aligned}$$

The first integral in the last expression is the normalization integral and therefore equals one:

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\,dx = 1\,.$$

Taking this into account and changing the integration variable in the second integral to $y = x - a$, we get

$$\langle x\rangle = a + \int_{-\infty}^{+\infty} f(y)\,dy\,,$$

where

$$f(y) = y|\Psi(a+y,t)|^2\,.$$

The fact that $|\Psi(x,t)|^2$ is symmetric about $x = a$ implies that the function $f(y)$ is odd. Indeed,

$$f(-y) = -y|\Psi(a-y,t)|^2 = -y|\Psi(a+y,t)|^2 = -f(y)\,.$$

Therefore, the integral of $f(y)$ is zero, $\int_{-\infty}^{+\infty} f(y)\,dy = 0$, and, consequently,

$$\langle x\rangle = a\,.$$