

## EXERCISE 12

### Classical limit in the particle-in-a-box system

Consider a particle confined to a one-dimensional box with walls at  $x = 0$  and  $x = L$ . Suppose the particle is in the  $n^{\text{th}}$  stationary state, described by the wave function

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right).$$

Calculate the probability  $P_n(a, b)$  of finding the particle within the interval  $a < x < b$ , where  $0 < a < b < L$ . Show that as  $n$  tends to infinity,  $P_n(a, b)$  approaches the classical-mechanical position-independent value  $(b - a)/L$ .

## Solution

The probability density describing the particle in the  $n^{\text{th}}$  stationary state is

$$|\psi_n(x)|^2 = \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right).$$

The probability of finding the particle within the interval  $a < x < b$  (provided  $0 < a < b < L$ ) is given by

$$\begin{aligned} P_n(a, b) &= \int_a^b |\psi_n(x)|^2 dx \\ &= \frac{1}{L} \int_a^b 2 \sin^2\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \int_a^b \left[1 - \cos\left(\frac{2n\pi x}{L}\right)\right] dx \\ &= \frac{1}{L} \left\{ (b - a) - \frac{L}{2n\pi} \sin\left(\frac{2n\pi x}{L}\right) \Big|_{x=a}^{x=b} \right\}, \end{aligned}$$

or

$$P_n(a, b) = \frac{b - a}{L} - \frac{\Delta}{n\pi}$$

where

$$\Delta \equiv \frac{1}{2} \left[ \sin\left(\frac{2n\pi b}{L}\right) - \sin\left(\frac{2n\pi a}{L}\right) \right].$$

For large  $n$ , the factor  $\Delta$  becomes highly sensitive to the values of  $a$  and  $b$ . In other words,  $\Delta$  can change rapidly even with slight variations in  $a$  or  $b$ . However, the value of  $\Delta$  is always bounded between  $-1$  and  $1$ :

$$-1 \leq \Delta \leq 1.$$

As  $n$  tends to infinity, the ratio  $\Delta/n$  approaches zero. Consequently, the probability density  $P_n$  converges to the uniform distribution:

$$P_n(a, b) \rightarrow \frac{b - a}{L} \quad \text{as} \quad n \rightarrow \infty.$$