Exercise 7
$\langle x \rangle$, $\langle p \rangle$, $\langle V \rangle$, and $\langle T \rangle$ for a coherent state

A particle undergoes harmonic oscillations within a potential well described by the potential function

$$V(x) = \frac{1}{2} m\omega^2 x^2 \,.$$

The oscillations are characterized by the coherent state wave function

$$\Psi(x,t) = \left(\frac{2\alpha}{\pi}\right)^{1/4} \exp\left(-\alpha(x - x_0 \cos\omega t)^2 - i2\alpha x x_0 \sin\omega t + i\frac{\alpha x_0^2}{2}\sin 2\omega t - i\frac{\omega t}{2}\right)$$

with

$$\alpha = \frac{m\omega}{2\hbar} \,.$$

Determine the time-dependent expectation values of the particle's position, momentum, potential energy, and kinetic energy.

Solution

The probability density corresponding to the coherent state is given by

$$|\Psi(x,t)|^2 = \sqrt{\frac{2\alpha}{\pi}} e^{-2\alpha(x-x_0\cos\omega t)^2} \, .$$

This function is symmetric about $x = x_0 \cos\omega t$. As it was shown in EXERCISE 3, the position expectation value of a symmetric probability distribution coincides with its symmetry point. Hence,

$$\boxed{\langle x \rangle = x_0 \cos\omega t}$$

To find $\langle p \rangle$ we first differentiate Ψ with respect to x:

$$\frac{\partial \Psi}{\partial x} = \left[-2\alpha(x - x_0\cos\omega t) - i2\alpha x_0 \sin\omega t\right] \Psi \, .$$

Then,

$$\begin{aligned}
\langle p \rangle &= -i\hbar \int_{-\infty}^{+\infty} \Psi^* \frac{\partial \Psi}{\partial x} \, dx \\
&= -i\hbar \int_{-\infty}^{+\infty} \Psi^* \left[-2\alpha(x - x_0\cos\omega t) - i2\alpha x_0 \sin\omega t\right] \Psi \\
&= 2\alpha\hbar \left\{ i \int_{-\infty}^{+\infty} x|\Psi|^2 \, dx - x_0(i\cos\omega t + \sin\omega t) \int_{-\infty}^{+\infty} |\Psi|^2 \, dx \right\} \, .
\end{aligned}$$

The first integral inside the curly brackets is the particle's mean position $\langle x \rangle$, which we've previously determined to be $x_0 \cos\omega t$. In accordance with the normalization condition, the second integral equals one. Thus,

$$\begin{aligned}
\langle p \rangle &= 2\alpha\hbar \Big\{ i x_0 \cos\omega t - x_0(i\cos\omega t + \sin\omega t) \Big\} \\
&= -2\alpha\hbar x_0 \sin\omega t \, .
\end{aligned}$$

Substituting $\alpha = m\omega/2\hbar$, we obtain

$$\boxed{\langle p \rangle = -m\omega x_0 \sin\omega t}$$

Let's also consider an alternative, much shorter path to the same result. Recalling the definition of the momentum expectation value, we have

$$\langle p \rangle = m\frac{d\langle x \rangle}{dt} = m\frac{d}{dt} x_0 \cos\omega t = -m\omega x_0 \sin\omega t \, .$$

The expectation value of the particle's potential energy is given by

$$\begin{aligned}\langle V\rangle &= \int_{-\infty}^{+\infty} \Psi^* \left(\frac{1}{2}m\omega^2 x^2\right) \Psi \, dx \\ &= \frac{m\omega^2}{2} \int_{-\infty}^{+\infty} x^2 |\Psi|^2 \, dx \\ &= \frac{m\omega^2}{2} \int_{-\infty}^{+\infty} x^2 \sqrt{\frac{2\alpha}{\pi}} e^{-2\alpha(x - x_0\cos\omega t)^2} \, dx \,.\end{aligned}$$

Changing the integration variable to $y = \sqrt{2\alpha}(x - x_0 \cos\omega t)$, we have

$$\begin{aligned}\langle V\rangle &= \frac{m\omega^2}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \left(\frac{y}{\sqrt{2\alpha}} + x_0\cos\omega t\right)^2 e^{-y^2}\, dy \\ &= \frac{m\omega^2}{2\sqrt{\pi}} \left\{ \frac{1}{2\alpha} \int_{-\infty}^{+\infty} y^2 e^{-y^2}\, dy + 2\frac{x_0\cos\omega t}{\sqrt{2\alpha}} \int_{-\infty}^{+\infty} y e^{-y^2}\, dy + x_0^2 \cos^2\omega t \int_{-\infty}^{+\infty} e^{-y^2}\, dy \right\} \,.\end{aligned}$$

Using

$$\int_{-\infty}^{+\infty} y^2 e^{-y^2}\, dy = \frac{\sqrt{\pi}}{2}\,, \qquad \int_{-\infty}^{+\infty} y e^{-y^2}\, dy = 0\,, \qquad \int_{-\infty}^{+\infty} e^{-y^2}\, dy = \sqrt{\pi}\,,$$

we obtain

$$\langle V\rangle = \frac{m\omega^2}{2}\left(\frac{1}{4\alpha} + x_0^2\cos^2\omega t\right) \,.$$

Then, substituting $\alpha = m\omega/2\hbar$, we end up with

$$\boxed{\langle V\rangle = \frac{m\omega^2}{2} x_0^2 \cos^2\omega t + \frac{\hbar\omega}{4}}$$

Note that $\langle V\rangle$ can be also written as

$$\langle V\rangle = \frac{m\omega^2}{2}\langle x\rangle^2 + \frac{\hbar\omega}{4} \,.$$

Finally, for the expectation value of the particle's kinetic energy, we have

$$\begin{aligned}\langle T\rangle &= \int_{-\infty}^{+\infty} \Psi^* \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right) \Psi\, dx \\ &= -\frac{\hbar^2}{2m} \int_{-\infty}^{+\infty} \Psi^* \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial x}\right) dx \,.\end{aligned}$$

Doing the last integral by parts,

$$\begin{aligned}\langle T\rangle &= -\frac{\hbar^2}{2m} \left\{ \Psi^* \frac{\partial\Psi}{\partial x}\bigg|_{x\to-\infty}^{x\to+\infty} - \int_{-\infty}^{+\infty} \frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x}\, dx \right\} \\ &= \frac{\hbar^2}{2m} \int_{-\infty}^{+\infty} \left|\frac{\partial\Psi}{\partial x}\right|^2 dx \,.\end{aligned}$$

In the last step, we made use of the fact that the function $\Psi^* \frac{\partial \Psi}{\partial x}$ vanishes as $x \to \pm\infty$. Now, using the expression for $\frac{\partial \Psi}{\partial x}$ we obtained earlier, we write

$$\begin{aligned}
\langle T \rangle &= \frac{\hbar^2}{2m} \int_{-\infty}^{+\infty} \Big| -2\alpha(x - x_0 \cos \omega t) - i2\alpha x_0 \sin \omega t \Big|^2 |\Psi|^2 \, dx \\
&= \frac{\hbar^2}{2m} (2\alpha)^2 \int_{-\infty}^{+\infty} \Big((x - x_0 \cos \omega t)^2 + x_0^2 \sin^2 \omega t \Big) |\Psi|^2 \, dx \\
&= \frac{2\alpha^2 \hbar^2}{m} \left\{ \int_{-\infty}^{+\infty} (x - x_0 \cos \omega t)^2 |\Psi|^2 \, dx + x_0^2 \sin^2 \omega t \int_{-\infty}^{+\infty} |\Psi|^2 \, dx \right\} .
\end{aligned}$$

Writing $|\Psi|^2$ explicitly in the first integral and recognizing that the second integral is the normalization integral, we obtain

$$\langle T \rangle = \frac{2\alpha^2 \hbar^2}{m} \left\{ \int_{-\infty}^{+\infty} (x - x_0 \cos \omega t)^2 \sqrt{\frac{2\alpha}{\pi}} e^{-2\alpha(x - x_0 \cos \omega t)^2} \, dx + x_0^2 \sin^2 \omega t \right\} .$$

The remaining integral can be calculated by changing the integration variable to $y = \sqrt{2\alpha}(x - x_0 \cos \omega t)$:

$$\begin{aligned}
\langle T \rangle &= \frac{2\alpha^2 \hbar^2}{m} \left\{ \frac{1}{2\alpha\sqrt{\pi}} \underbrace{\int_{-\infty}^{+\infty} y^2 e^{-y^2} \, dy}_{\sqrt{\pi}/2} + x_0^2 \sin^2 \omega t \right\} \\
&= \frac{2\alpha^2 \hbar^2}{m} \left\{ \frac{1}{4\alpha} + x_0^2 \sin^2 \omega t \right\} .
\end{aligned}$$

Substituting $\alpha = m\omega/2\hbar$, we arrive at

$$\boxed{\langle T \rangle = \frac{m\omega^2}{2} x_0^2 \sin^2 \omega t + \frac{\hbar \omega}{4}}$$

Note that $\langle T \rangle$ can be also written as

$$\langle T \rangle = \frac{\langle p \rangle^2}{2m} + \frac{\hbar \omega}{4} .$$