Exercise 15

An alternative form of the normalization condition

Any wave function $\Psi(x,t)$ of a quantum system governed by a Hamiltonian $\hat{H}$ can be represented as a superposition of the system's stationary states:

$$\Psi(x,t) = c_1\psi_1(x)e^{-iE_1t/\hbar} + c_2\psi_2(x)e^{-iE_2t/\hbar} + c_3\psi_2(x)e^{-iE_3t/\hbar} + \ldots$$

Here, ψ_n's are the spatial components of the stationary states, satisfying the time-independent Schrödinger equation

$$\hat{H}\psi_n = E_n\psi_n \qquad (n = 1, 2, 3, \ldots)$$

and the normalization condition

$$\int_{-\infty}^{+\infty} |\psi_n(x)|^2 \, dx = 1 \qquad (n = 1, 2, 3, \ldots)\,,$$

and c_n's are (generally complex) expansion coefficients. For simplicity, let's assume that the stationary state energies are all different and arranged in the ascending order:

$$E_1 < E_2 < E_3 < \ldots$$

Show that the requirement for $\Psi(x,t)$ to satisfy the normalization condition

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx = 1$$

is equivalent to the requirement that the sum of the squared moduli of the expansion coefficients equals one, i.e.

$$|c_1|^2 + |c_2|^2 + |c_3|^2 + \ldots = 1\,.$$

Solution

The probability density characterizing the superposition state

$$\begin{aligned}\Psi(x,t) &= c_1\psi_1(x)e^{-iE_1t/\hbar} + c_2\psi_2(x)e^{-iE_2t/\hbar} + c_3\psi_2(x)e^{-iE_3t/\hbar} + \ldots \\ &= \sum_n c_n\psi_n(x)e^{-iE_nt/\hbar}\,,\end{aligned}$$

where the sum runs over all stationary states of the system, is given by

$$\begin{aligned}|\Psi(x,t)|^2 &= \Psi^*(x,t)\Psi(x,t) \\ &= \left(\sum_k c_k\psi_k(x)e^{-iE_kt/\hbar}\right)^* \sum_n c_n\psi_n(x)e^{-iE_nt/\hbar} \\ &= \sum_k c_k^*\psi_k^*(x)e^{iE_kt/\hbar}\sum_n c_n\psi_n(x)e^{-iE_nt/\hbar} \\ &= \sum_k\sum_n c_k^*c_n e^{i(E_k-E_n)t/\hbar}\psi_k^*(x)\psi_n(x)\,.\end{aligned}$$

The double sum on the right-hand side can be separated into two contributions: a "diagonal" part including terms with $k = n$ and an "off-diagonal" part summing terms with $k \neq n$, i.e.

$$|\Psi(x,t)|^2 = \underbrace{\sum_n |c_n|^2|\psi_n(x)|^2}_{\text{diagonal}} + \underbrace{\sum_k\sum_{n\neq k} c_k^*c_n e^{i(E_k-E_n)t/\hbar}\psi_k^*(x)\psi_n(x)}_{\text{off-diagonal}}\,.$$

Then, the normalization integral can be written as

$$\begin{aligned}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx &= \sum_n |c_n|^2\int_{-\infty}^{+\infty}|\psi_n(x)|^2\,dx \\ &\quad + \sum_k\sum_{n\neq k} c_k^*c_n e^{i(E_k-E_n)t/\hbar}\int_{-\infty}^{+\infty}\psi_k^*(x)\psi_n(x)\,dx\,.\end{aligned}$$

The first integral in the right-hand side equals one in view the normalization condition satisfied by all stationary states:

$$\int_{-\infty}^{+\infty}|\psi_n(x)|^2\,dx = 1\,.$$

The second integral equals zero due to the orthogonality condition satisfied by stationary states with different energies (see Exercise 14):

$$\int_{-\infty}^{+\infty}\psi_k^*(x)\psi_n(x)\,dx = 0 \qquad (k \neq n)\,.$$

Hence, the above expression for the integrated probability density simplifies to

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx = \sum_n |c_n|^2 \, .$$

This implies that the requirement for $\Psi(x,t)$ to be normalized to unity can be written in the following form:

$$\boxed{\sum_n |c_n|^2 = |c_1|^2 + |c_2|^2 + |c_3|^2 + \ldots = 1}$$