Exercise 2
An example of using the Born rule

The time-dependent probability density of a particle performing harmonic oscillations is given by

$$|\Psi(x,t)|^2 = \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left(-\frac{m\omega}{\hbar}(x - x_0\cos\omega t)^2\right) .$$

Find the probability $P(t)$ to find the particle inside the spatial interval

$$-\sqrt{\frac{\hbar}{m\omega}} < x < \sqrt{\frac{\hbar}{m\omega}} .$$

Sketch and discuss the function $P(t)$ in the following two parametric regimes:

(a)

$$0 < x_0 \ll \sqrt{\frac{\hbar}{m\omega}} ,$$

(b)

$$x_0 \gg \sqrt{\frac{\hbar}{m\omega}} .$$

Solution

Let's define a length scale a as

$$\boxed{a = \sqrt{\frac{\hbar}{m\omega}}}$$

Then, the probability of finding the particle between $-a$ and a is given by

$$P(t) = \int_{-a}^{a} |\Psi(x,t)|^2 \, dx = \frac{1}{\sqrt{\pi}a} \int_{-a}^{a} e^{-(x-x_0\cos\omega t)^2/a^2} dx = \frac{1}{\sqrt{\pi}} \int_{-1-\frac{x_0}{a}\cos\omega t}^{1-\frac{x_0}{a}\cos\omega t} e^{-u^2} du \, .$$

Using that the fact that the function e^{-u^2} is even, $P(t)$ can be rewritten as follows:

$$\begin{aligned} P(t) &= \frac{1}{\sqrt{\pi}} \int_0^{1-\frac{x_0}{a}\cos\omega t} e^{-u^2} du + \frac{1}{\sqrt{\pi}} \int_{-1-\frac{x_0}{a}\cos\omega t}^{0} e^{-u^2} du \\ &= \frac{1}{\sqrt{\pi}} \int_0^{1-\frac{x_0}{a}\cos\omega t} e^{-u^2} du + \frac{1}{\sqrt{\pi}} \int_0^{1+\frac{x_0}{a}\cos\omega t} e^{-u^2} du \, . \end{aligned}$$

In terms of the error function

$$\operatorname{erf} z \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du \, ,$$

we arrive at the following expression for the probability:

$$\boxed{P(t) = \frac{1}{2} \operatorname{erf}\left(1 - \frac{x_0}{a}\cos\omega t\right) + \frac{1}{2} \operatorname{erf}\left(1 + \frac{x_0}{a}\cos\omega t\right)}$$

Now, let's consider the following two parametric regimes.

(a) If $0 < x_0 \ll a$, the quantity $\epsilon = \frac{x_0}{a}\cos\omega t$ is small compared to 1. Hence, we can use the Taylor approximation of $P(t)$ around $\epsilon = 0$. We have

$$\begin{aligned} P &\simeq \frac{1}{2}\left(\operatorname{erf} 1 - \frac{2\epsilon}{e\sqrt{\pi}} - \frac{2\epsilon^2}{e\sqrt{\pi}} + \dots\right) + \frac{1}{2}\left(\operatorname{erf} 1 + \frac{2\epsilon}{e\sqrt{\pi}} - \frac{2\epsilon^2}{e\sqrt{\pi}} + \dots\right) \\ &\simeq \operatorname{erf} 1 - \frac{2\epsilon^2}{e\sqrt{\pi}} \, , \end{aligned}$$

or

$$P(t) \simeq \operatorname{erf} 1 - \frac{2}{e\sqrt{\pi}} \left(\frac{x_0}{a}\right)^2 \cos^2\omega t \, .$$

The last expression shows that $P(t)$ exhibits simple sinusoidal oscillations in time.

In this parametric regime, a significant portion of the wave packet remains consistently confined within the region between $-a$ and a. Only the tails of the wave packet extend beyond this region, periodically leaving and re-entering it, leading to sinusoidal oscillations in the probability function. Figures 1 and 2 illustrate the function $P(t)$ for the case where $x_0/a = 0.1$.

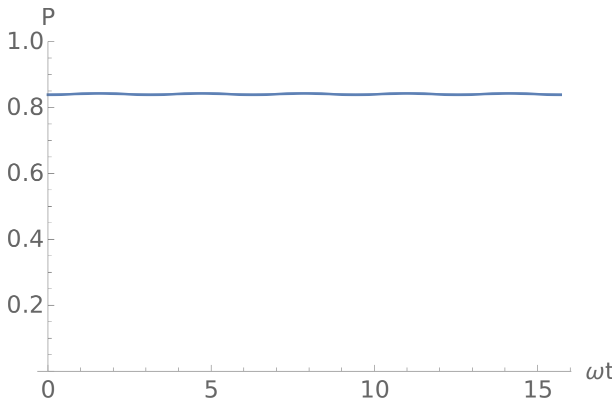

Figure 1: Probability P as a function of ωt for $x_0/a = 0.1$.

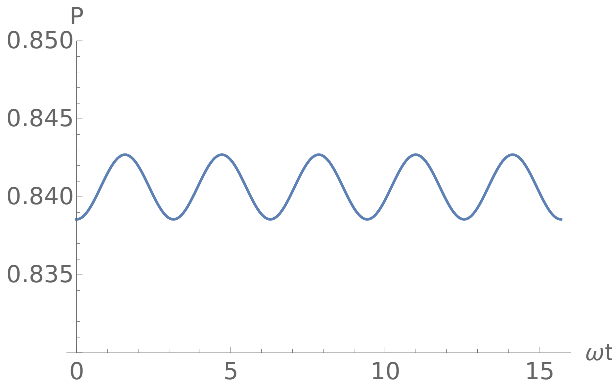

Figure 2: A close-up of Figure 1.

(b) If $x_0 \gg a$, the wave packet starts off outside the interval between $-a$ and a. As time progresses, its center oscillates within the parabolic potential well between the turning points at $-x_0$ and x_0. Most of the time, the wave packet remains outside the $(-a, a)$ interval, causing $P(t)$ to be zero for the majority of the time. Only during the brief moments when the wave packet passes through the origin does $P(t)$ exhibit nonzero values. Figure 3 illustrates the time dependence of P for the case where $x_0/a = 10$.

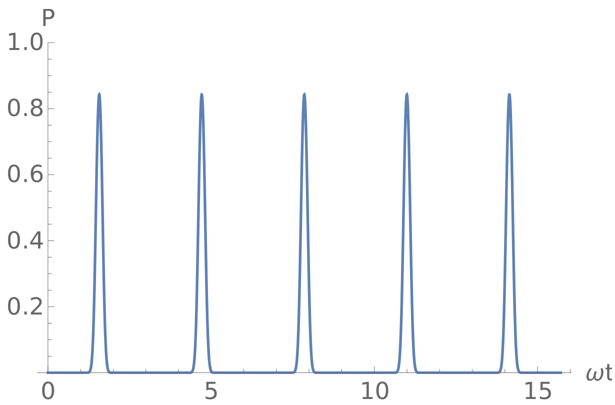

Figure 3: Probability P as a function of ωt for $x_0/a = 10$.