Exercise 6
Mean momentum of a particle with a real wave function

Show that if a particle's wave function $\Psi(x,t)$ is real at a particular time t, then the expectation value of the particle's momentum at that time is zero.

Solution

The expectation value of the particle's momentum is given by

$$\langle p \rangle = -i\hbar \int_{-\infty}^{+\infty} \Psi^* \frac{\partial \Psi}{\partial x} \, dx \, . \tag{1}$$

Doing the last integral by parts, we have

$$\langle p \rangle = -i\hbar \left(\Psi^* \Psi \Big|_{x \to -\infty}^{x \to +\infty} - \int_{-\infty}^{+\infty} \frac{\partial \Psi^*}{\partial x} \Psi \, dx \right) \, .$$

As the probability density $\Psi^* \Psi = |\Psi|^2$ decays to zero as $x \to \pm\infty$, the first term inside the parentheses vanishes. Hence,

$$\langle p \rangle = i\hbar \int_{-\infty}^{+\infty} \Psi \frac{\partial \Psi^*}{\partial x} \, dx \, . \tag{2}$$

Equations (1) and (2) hold true for any wave function Ψ.

Now, let us take into account that, as stated in the problem, Ψ is real-valued, i.e.

$$\Psi^* = \Psi \, .$$

Then, Eq. (1) becomes

$$\langle p \rangle = -i\hbar \int_{-\infty}^{+\infty} \Psi \frac{\partial \Psi}{\partial x} \, dx \, ,$$

while Eq. (2) reads

$$\langle p \rangle = i\hbar \int_{-\infty}^{+\infty} \Psi \frac{\partial \Psi}{\partial x} \, dx \, .$$

Comparing the last two equations, we find that

$$\langle p \rangle = -\langle p \rangle \, ,$$

or

$$\boxed{\langle p \rangle = 0}$$