

EXERCISE 13

Temporal oscillations in a superposition of two stationary states

A particle of mass m confined to a box of width L is in a state Ψ , represented by a superposition of the ground state ($n = 1$) and first excited state ($n = 2$):

$$\Psi(x, t) = \frac{1}{\sqrt{2}} \left(\psi_1(x) e^{-iE_1 t/\hbar} + \psi_2(x) e^{-iE_2 t/\hbar} \right),$$

where

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

is the spatial part of the n^{th} stationary state, and

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

is the corresponding stationary state energy.

1. Show that the probability density of the particle is given by

$$|\Psi(x, t)|^2 = \frac{\psi_1(x)^2}{2} + \frac{\psi_2(x)^2}{2} + \psi_1(x)\psi_2(x) \cos\left(\frac{E_2 - E_1}{\hbar} t\right).$$

2. Suppose the particle is an electron ($m = 9.11 \times 10^{-31}$ kg) and the width of the box is $L = 1$ nm. What are the frequency and period of the temporal oscillations in the probability density?
3. Verify that $\Psi(x, t)$ is normalized to unity.

Solution

1. Using the fact that the functions ψ_n are real for all n , we write

$$\begin{aligned}
 |\Psi(x)|^2 &= \Psi^*(x)\Psi(x) \\
 &= \frac{1}{\sqrt{2}} \left(\psi_1(x)e^{iE_1t/\hbar} + \psi_2(x)e^{iE_2t/\hbar} \right) \\
 &\quad \times \frac{1}{\sqrt{2}} \left(\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar} \right) \\
 &= \frac{1}{2} \left[\psi_1(x)^2 + \psi_2(x)^2 + \psi_1(x)\psi_2(x) \underbrace{\left(e^{i(E_1-E_2)t/\hbar} + e^{i(E_2-E_1)t/\hbar} \right)}_{2 \cos \left[(E_2-E_1)t/\hbar \right]} \right],
 \end{aligned}$$

or

$$\boxed{|\Psi(x, t)|^2 = \frac{\psi_1(x)^2}{2} + \frac{\psi_2(x)^2}{2} + \psi_1(x)\psi_2(x) \cos \left(\frac{E_2 - E_1}{\hbar} t \right)}$$

2. The oscillation frequency of the probability density is given by

$$\begin{aligned}
 \omega &= \frac{E_2 - E_1}{\hbar} \\
 &= \frac{1}{\hbar} \left(\frac{2^2 \pi^2 \hbar^2}{2mL^2} - \frac{1^2 \pi^2 \hbar^2}{2mL^2} \right) \\
 &= \frac{3\pi^2 \hbar}{2mL^2}.
 \end{aligned}$$

Substituting $\hbar = 1.05 \times 10^{-34} \text{ kg m}^2 \text{ s}^{-1}$, $m = 9.11 \times 10^{-31} \text{ kg}$, and $L = 10^{-9} \text{ m}$, we get

$$\omega = 1.71 \times 10^{15} \text{ s}^{-1} = 1.71 \text{ PHZ}.$$

Then, the oscillation period is

$$T = \frac{2\pi}{\omega} = 3.68 \times 10^{-15} \text{ s} = 3.68 \text{ fs}.$$

3. For the normalization integral, we have

$$\begin{aligned}
 \int_0^L |\Psi(x)|^2 dx &= \frac{1}{2} \int_0^L \psi_1(x)^2 dx + \frac{1}{2} \int_0^L \psi_2(x)^2 dx \\
 &\quad + \cos \left(\frac{E_2 - E_1}{\hbar} t \right) \int_0^L \psi_1(x)\psi_2(x) dx.
 \end{aligned}$$

The first two integrals equal one, as they are the normalization integrals for the ground state and the first excited state. Hence,

$$\int_0^L |\Psi(x)|^2 dx = 1 + \cos\left(\frac{E_2 - E_1}{\hbar}t\right) \int_0^L \psi_1(x)\psi_2(x) dx.$$

Then, since

$$\begin{aligned} \int_0^L \psi_1(x)\psi_2(x) dx &= \frac{2}{L} \int_0^L \underbrace{\sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right)}_{\frac{1}{2} \cos\left(\frac{2\pi x}{L} - \frac{\pi x}{L}\right) - \frac{1}{2} \cos\left(\frac{2\pi x}{L} + \frac{\pi x}{L}\right)} dx \\ &= \frac{1}{L} \int_0^L \cos\left(\frac{\pi x}{L}\right) dx - \frac{1}{L} \int_0^L \cos\left(\frac{3\pi x}{L}\right) dx \\ &= \frac{1}{\pi} \underbrace{\sin\left(\frac{\pi x}{L}\right)\Big|_{x=0}^{x=L}}_0 - \frac{1}{3\pi} \underbrace{\sin\left(\frac{3\pi x}{L}\right)\Big|_{x=0}^{x=L}}_0 \\ &= 0, \end{aligned}$$

we end up with

$$\boxed{\int_0^L |\Psi(x)|^2 dx = 1}$$